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# Analysis of fatigue data for lifetime predictions for ceramic materials

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Accuracy in data analysis is of utmost importance because lifetime predictions are extremely sensitive to experimental uncertainty in the crack growth parameters. The limitations of the conventional data reduction techniques used for analysing static and dynamic fatigue data are reviewed and new, statistical methods of data reduction that offer advantages over the conventional techniques are discussed.

## 1. Introduction

The use of ceramic materials in critical, high strength applications motivated the development of analytical techniques for predicting the lifetime and reliability of ceramic structural components [1–3]. These predictive techniques are based on the reasonable assumption that failure of ceramic materials occurs predominantly from the stress-dependent growth of a pre-existing flaw to dimensions critical for spontaneous crack propagation. For most ceramics subcritical crack velocity ($V$) can be expressed as a power function of the stress intensity factor ($K_I$) [4]:

$$V = AK_I^N \tag{1}$$

where $A$ and $N$ are constants that depend on the environment and material composition. From Equation 1 it can be derived that the time to failure ($t_f$) under a constant tensile stress ($\sigma_a$) is [1–3]:

$$t_f = BS_i^{N-2}\sigma_a^{-N} \tag{2}$$

where $B = 2/(AY^2(N-2)\,K_{IC}^{N-2})$, $Y$ = geometric constant ($\sqrt{\pi}$ for surface flaws), $K_{IC}$ = critical stress intensity factor, and $S_i$ = fracture strength in an inert environment. The time to failure in Equation 2 is simply the time required for a flaw to grow from an initial, subcritical length to the critical length for catastrophic crack propagation. $B$ and $N$ are the fatigue constants that characterize this subcritical crack growth. The initial flaw size is characterized in Equation 2 by the fracture strength of the component in an inert environment. Since the distribution of initial flaw sizes is directly related to the distribution of inert strengths, the probability of failure ($F$) for a given $t_f$ and $\sigma_a$ can be obtained from Equation 2 by replacing the inert strength with its corresponding failure probability. By so doing, it is assumed that the origin of fracture is the same for both fatigue and inert failures and that the sample with the shortest fatigue life has the lowest inert strength and the sample with the median life has the median inert strength, etc.

It can also be shown that the minimum lifetime after proof testing ($t_{min}$), assuming no crack growth on unloading, is given by [1–3]:

$$t_{min} = B\sigma_p^{N-2}\sigma_a^{-N} \tag{3}$$

where $\sigma_p$ = proof stress. From a fracture mechanics viewpoint, the value of proof testing is that it characterizes the largest effective flaw possible in the tested component since any larger flaw would have caused failure during the proof test. Thus, $t_{min}$ is then the time it takes for this maximum length flaw to grow under a constant applied stress to critical dimensions for spontaneous fracture.

From Equation 2 and 3 it is seen that lifetime predictions for ceramic materials are dependent on the applied stress, the inert strength of the actual components, and the experimental parameters $B$ and $N$ that characterize subcritical crack growth. These crack growth parameters must be measured under conditions representative of the service environment. They can be obtained from crack velocity [4], static fatigue, or dynamic fatigue experiments [3, 5]. Since lifetime predictions are extremely sensitive to experimental uncertainty in the crack growth parameters [6, 7], accuracy in data analysis is of utmost importance. The purpose of this paper is to review and delineate the limitations of the conventional data reduction techniques used for analysing static and dynamic fatigue data and to introduce new, statistical methods of data reduction that offer advantages over the conventional techniques. Emphasis has been placed on data from static and dynamic tests because analysis of crack velocity data is relatively straightforward. In addition, for purposes of failure prediction data from static and dynamic fatigue tests are more reliable since data from large, preformed cracks may not be relevant to the propagation of the microscopic cracks present in structural ceramics [8, 9]. Also lifetime predictions based on crack velocity data generally involve greater extrapolation than do those based on static or dynamic fatigue data [10].

## 2. Analysis

Before discussing the various fatigue data analyses, it is important to note the basic problem in determining the crack growth parameters $B$ and $N$ by the static and dynamic fatigue test techniques. That is, one cannot measure on the same sample both the fatigue behaviour and the size of the initial pre-existing flaw that causes fracture. This is true because non-destructive techniques are not currently available to measure directly the microscopic flaw sizes present in structural ceramics. However, one means that the initial flaw size distribution can be estimated is by measuring the inert strengths on samples statistically similar to those used in the fatigue experiments. Then by pairing the fatigue and inert strength data at equal failure probabilities, the initial flaw size in a particular fatigue sample can be estimated. The various data analysis techniques discussed below differ basically in how the fatigue data is paired with the inert strength data.

### 2.1. Static fatigue data

Static fatigue data is generally generated by measuring the time to failure of a large number of samples at several constant applied stresses. From this data the median value of $t_f$ can be determined as a function of $\sigma_a$. By measuring the median $S_i$ on a group of statistically identical samples, $B$ and $N$ can be determined if Equation 2 is rewritten in terms of the median values of $t_f$ and $S_i$ as:

$$\ln (t_f)_{0.5} = \ln B + (N-2) \ln (S_i)_{0.5} - N \ln \sigma_a \tag{4}$$

where $(t_f)_{0.5}$ and $(S_i)_{0.5}$ = median values of $t_f$ and $S_i$. From a regression analysis of $\ln (t_f)_{0.5}$ versus $\ln \sigma_a$, $N$ and $B$ are calculated from:

$$\begin{aligned} \text{slope} &= -N \\ \text{intercept} &= \ln B + (N-2) \ln (S_i)_{0.5} \end{aligned} \tag{5}$$

The median values of $t_f$ and $S_i$ are used since these values are at equal failure probabilities and since it is the median value that is best estimated from relatively small sample sizes. Although the median value technique for analysing static fatigue data is quite straightforward to apply, it makes inefficient use of the data since only the median $t_f$ values are used in the regression analysis; thus, the uncertainty in $B$ and $N$ can be large.

Recognizing the need for more efficient utilization of fatigue data, several researchers [11, 12] have suggested a method of data reduction that is based on a homologous stress ratio ($\sigma_{HS}$), defined as

$$\sigma_{HS} = \frac{\sigma_a}{S_i} \tag{6}$$

To obtain fatigue data in terms of $\sigma_{HS}$, $t_f$ values for each applied stress are ranked and then paired with equal ranked $S_i$ values, i.e., the lowest $S_i$ value is paired with the shortest $t_f$, the next lowest $S_i$ with the next shortest $t_f$ and so on. Since $\sigma_a$ is fixed for a given ranking of fatigue lives, the relationship between $\sigma_{HS}$, and $t_f$ is established. In terms of the homologous stress ratio, Equation 2 can be rewritten as:

$$\ln (t_f S_i^2) = \ln B - N \ln \sigma_{HS} \tag{7}$$

From a regression analysis of $\ln (t_f S_i^2)$ versus $\ln \sigma_{HS}$, the constants $N$ and $B$ are determined from the slope and intercept, respectively. Since the data for all samples that fail in a measurable time are used in the regression analysis, this method greatly increases the confidence in $B$ and $N$ as compared to the median value method.

With both the median value and homologous stress technique, it is necessary to rank the $t_f$ data separately at each applied stress; thus, one needs to test a relatively large number of samples at each stress condition to determine an accurate estimate of the $t_f$ distribution. Considerable improvement in the statistics can be realized if all the data from every stress condition are ranked together. By rewriting Equation 2 as:

$$(\ln t_f + N \ln \sigma_a) = \ln B + (N-2) \ln S_i \tag{8}$$

it is seen that the left-hand-side of Equation 8 is directly related to $\ln S_i$; hence, the ranking of the data in terms of $(\ln t_f + N \ln \sigma_a)$ which will be defined as the quantity $R$, is equivalent to the ranking of $\ln S_i$. That is to say, the smallest value of $R$ corresponds to the sample that has the lowest inert strength, and so on. Therefore, by ranking the data in terms of $R$ and pairing these ranked values with equal ranked $\ln S_i$ values, the constants $N$ and $B$ can be determined from a regression analysis where the slope gives $N$ and intercept gives $B$. Although this procedure is relatively straightforward, a complication does arise because the value of $N$, necessary for the calculation of $R$, is not known *a priori*. Therefore, this data reduction technique must be used in an iterative scheme where an initial $N$ is assumed for the calculation of the values of $R$, which are then ranked and paired with equal ranked $\ln S_i$ values. A regression analysis of this data then yields a new value of $N$. This new value of $N$ is used to re-calculate and re-rank the values of $R$ and the regression is iterated until convergence on the value of $N$ is accomplished. At this convergence the value of $B$ is then determined. This data analysis technique will be referred to in this paper as the iterative, bivariant technique.

All of the three data analysis techniques discussed above make use of a known inert strength distribution. In some cases the inert strength distribution may not be known; however, static fatigue data can still be used to make failure predictions in the fracture mechanics framework since this data contains information about the fatigue behaviour as well as the initial flaw distribution. By assuming that the initial inert strength distribution (or equivalently, the initial flaw distribution) is given by the Weibull function [2],

$$\ln\left(\ln \frac{1}{1-F}\right) = m \ln \frac{S_i}{S_0} \tag{9}$$

where $F$ = cumulative failure probability and $m$, $S_0$ = constants, Equation 8 can be written:

$$(\ln t_f + N \ln \sigma_a) = C + \left(\frac{N-2}{m}\right) \ln\left(\ln \frac{1}{1-F}\right) \tag{10}$$

where $C = [\ln B + (N-2) \ln S_0]$. To determine the constants $N$, $C$, and $m$, values of $(\ln t_f + N \ln \sigma_a)$ are ranked assuming an initial $N$. Then by rewriting Equation 10 as:

$$\ln t_f = C + \left(\frac{N-2}{m}\right) \ln\left(\ln \frac{1}{1-F}\right) - N \ln \sigma_a \tag{11}$$

a trivariant regression analysis [13] can be used to determine $N$, $C$, and $(N - 2/m)$ from which $m$ is obtained. This "regressed" value of $N$ is then used to recalculate and re-rank the $(\ln t_f + n \ln \sigma_a)$ values and the trivariant regression is iterated until convergence on $N$ is obtained. The constants $C$ and $m$ are then determined at this convergence. This technique will be referred to as the iterative trivariant technique.

## 2.2. Dynamic fatigue data

Evans [14] has derived from Equation 1 that fracture strength ($S$) is related to the stress rate ($\dot{\sigma}$) by

$$S^{N+1} = B(N+1) S_i^{N-2} \dot{\sigma} \tag{12}$$

Since Equation 12 is of the same form as Equation 2, the analysis of dynamic fatigue data ($S$ versus $\dot{\sigma}$) can be carried out in a manner similar to that of static fatigue data. For example, Equation 12 can be rewritten in terms of median values of $S$ and $S_i$ as

$$\ln (S)_{0.5} = \frac{1}{N+1} [\ln B + \ln (N+1) + (N-2) \ln (S_i)_{0.5} + \ln \dot{\sigma}] \tag{13}$$

From a regression analysis of $\ln (S)_{0.5}$ versus $\ln \dot{\sigma}$, $N$ and $B$ are calculated from:

$$\begin{aligned} \text{slope} &= \frac{1}{N+1} \\ \text{intercept} &= \frac{1}{N+1} [\ln B + \ln (N+1) + (N-2) \ln (S_i)_{0.5}] \end{aligned} \tag{14}$$

Similarly, by defining a homologous stress ratio to be

$$\sigma_{HD} = S/S_i \tag{15}$$

Equation 12 can be rewritten

$$\ln \sigma_{HD} = \frac{1}{N+1} [\ln B + \ln (N+1) + \ln (\dot{\sigma}/S_i^3)] \tag{16}$$

By ranking the $S$ data for a given $\dot{\sigma}$ and then pairing this data with equal ranked $S_i$ data, a regression analysis of $\ln \sigma_{HD}$ versus $\ln(\dot{\sigma}/S_i^3)$ will give $N$ and $B$ from the slope and intercept, respectively.

$N$ and $B$ can also be determined from the iterative, bivariant technique by rewriting Equation 12 in the form

$$\left[\ln S - \left(\frac{1}{N+1}\right)\ln \dot{\sigma}\right] \qquad (17)$$

$$= \frac{1}{N+1}\left[\ln B + \ln(N+1) + (N-2)\ln S_i\right]$$

The iterative, bivariant regression analysis is identical to that for static fatigue data except now the quantity to be ranked is $[\ln S - (N+1)^{-1}\ln \dot{\sigma}]$. Similarly, $N$ and $B$ can be determined from the iterative, trivariant technique by substituting Equation 9 into Equation 17 to give

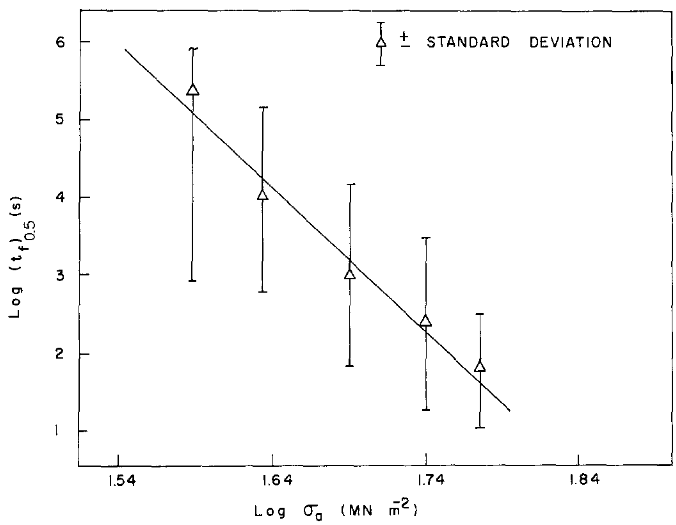
$$\left[\ln S - \left(\frac{1}{N+1}\right)\ln \dot{\sigma}\right] \qquad (18)$$

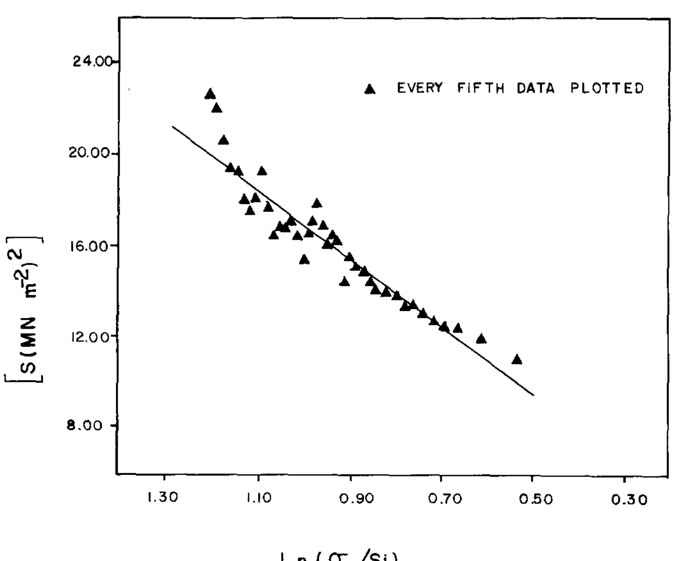
$$= \frac{1}{N+1}\left[D + \left(\frac{N-2}{m}\right)\ln\left(\ln\frac{1}{1-F}\right)\right]$$

Where $D = \ln B + \ln(N+1) + (N-2)\ln S_0$. By ranking the quantity on the left-hand-side of Equation 18, a trivariant analysis can be carried out by rearranging Equation 18 in the form:

$$\ln S = \frac{1}{N+1}\left[D + \left(\frac{N-2}{m}\right)\ln\left(\ln\frac{1}{1+F}\right) + \ln \dot{\sigma}\right] \qquad (19)$$

*Figure 1* Median time-to-failure as a function of applied stress for soda-lime glass tested in water at room temperature. The regression line has a correlation coefficient of 0.93.




lime glass tested in water at room temperature as analysed by the homologous stress technique. The regression line has a correlation coefficient of 0.94.

By iterating the trivariant regression until convergence on $N$ is obtained, the constants $N$, $D$, and $m$ are determined.

## 3. Application and discussion

To illustrate the application of the data reduction techniques described in the previous section, static and dynamic fatigue data [14] for soda-lime glass tested in water at room temperature was analysed. In the static fatigue tests, 261 samples were tested at 5 different applied stresses and in the dynamic fatigue tests, 180 samples were tested at 7 different stressing rates. The inert strength of 73 samples was measured in liquid nitrogen and the data fitted to the Weibull function to give:

$$\ln\left(\ln \frac{1}{1-F}\right) = 8.19 \ln (S_i/137.96) \quad (20)$$

where $S_i$ is in MPa. The median inert strength of the samples was 129.49 MPa. It should be noted that the Weibull distribution was used since it gave a good fit to the inert strength data: however, other statistical distributions could be used if they are shown more applicable.

Figs. 1 to 4 give the results of the regression analyses of the static fatigue data according to Equations 4, 7, 8, and 11, respectively. Figs. 5 to 8 show the results of the dynamic fatigue data

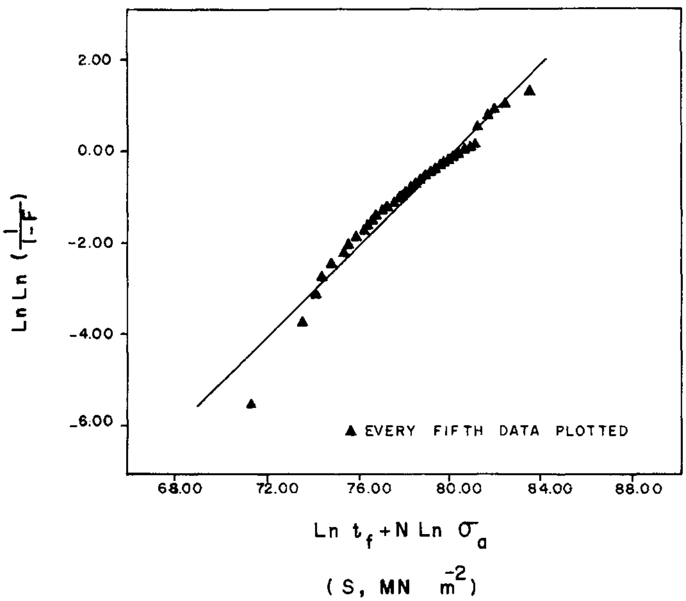



*Figure 3* Static fatigue data of soda-lime glass tested in water at room temperature as analysed by the iterative, bivariant technique. The regression line has a correlation coefficient of 0.98.

$\mathrm{Ln}\left[t_f (S_i)^2\right]$

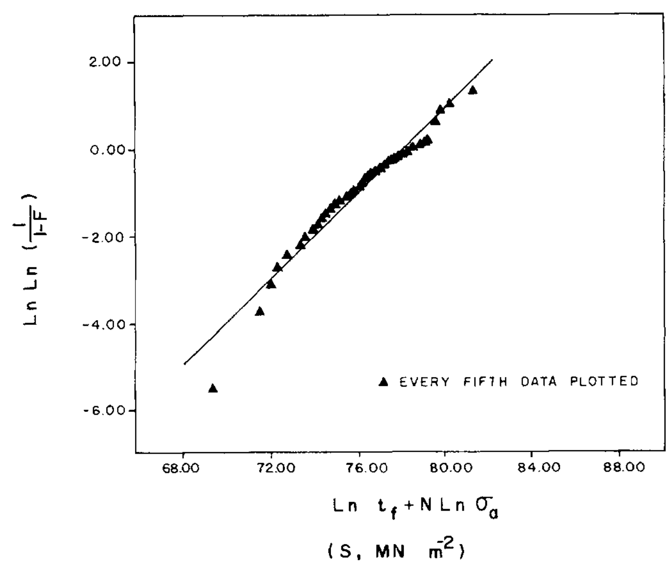



*Figure 4* Static fatigue data of soda-lime glass tested in water at room temperature as analysed by the iterative, trivariant technique. The regression line has a correlation coefficient of 0.98.

TABLE I Summary crack growth parameters for soda-lime glass in water

| Test data | Data analysis | $N$ | $\ln B$ [(MN m$^{-2}$)$^2$ sec] |
|---|---|---|---|
| Static fatigue | Median value | 19.32 (3.10)* | −1.66 (3.11) |
| Static fatigue | Homologous stress | 15.12 ( .38) | 1.82 ( .36) |
| Static fatigue | Iterative bivariant | 18.59 ( .22) | −1.37 ( .22) |
| Dynamic fatigue | Median value | 18.42 ( .79) | −1.70 ( .59) |
| Dynamic fatigue | Homologous stress | 17.96 ( .39) | −1.78 ( .28) |
| Dynamic fatigue | Iterative bivariant | 18.34 ( .14) | −1.99 ( .13) |

*The numbers in parentheses represents the standard deviation.

TABLE II Summary of fatigue parameters for soda-lime glass in water using iterative, trivariant data analysis

| Test data | $N$ | $m$ | $\ln B^{\dagger}$ [(MN m$^{-2}$)$^2$ sec] |
|---|---|---|---|
| Static fatigue | 18.05 (.26)* | 7.88 (.12) | −0.78 (.27) |
| Dynamic fatigue | 17.85 (.13) | 7.95 (.06) | −1.61 (.12) |

*The numbers in parentheses represent the standard deviation.

†This was calculated from the fitted parameters $C$ and $D$ of the static and dynamic fatigue data, respectively, using the value of $S_0 = 137.96$ MN m$^{-2}$.

analyses according to Equations 13, 16, 17, and 19, respectively. For making comparisons between data easier, the data as analysed by the iterative, bivariant techniques are plotted using the same axes as that used by the iterative, trivariant technique. Tables I and II summarize the crack growth parameters as determined from the various data analyses.

The general improvement of the fit to the data with the iterative bivariant and trivariant techniques, as compared to the median value and homologous stress techniques, is clearly evident in the figures and is undoubtedly related to the fact that the iterative techniques analyse all the data together rather than ranking the data separately for each applied stress or stressing rate condition. From Tables I and II it is seen that the values obtained for $N$ and $B$ are all similar and, hence, do not depend on the data analysis technique or on the particular experiment (static or dynamic fatigue) used. This similarity in the fatigue constants as determined from the various data analysis techniques was expected since a large enough number of samples was tested to give an accurate representation of the time-to-failure and strength distributions; hence, the good agreement between the median value and homologous stress techniques and the iterative techniques. The general agreement between the iterative techniques can also be seen by noting the similarity in the regression lines through the data in Figs. 3 and 4 and Figs. 7 and 8. It is also evident from Tables I and II that the techniques that utilize all of the data in the regression analysis (homologous stress, iterative bivariant, and iterative trivariant) give much better confidence in the fatigue constants as

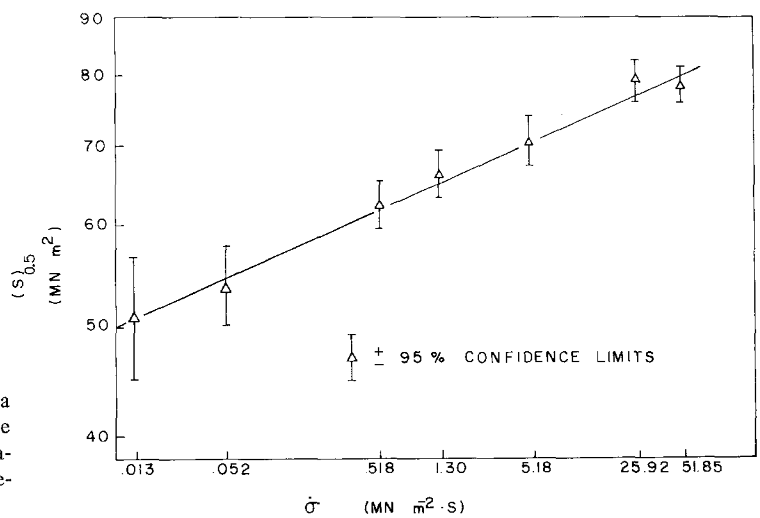



*Figure 5* Median fracture strength as a function of stressing rate for soda-lime glass tested in water at room temperature. The regression line has a correlation coefficient of 0.99.

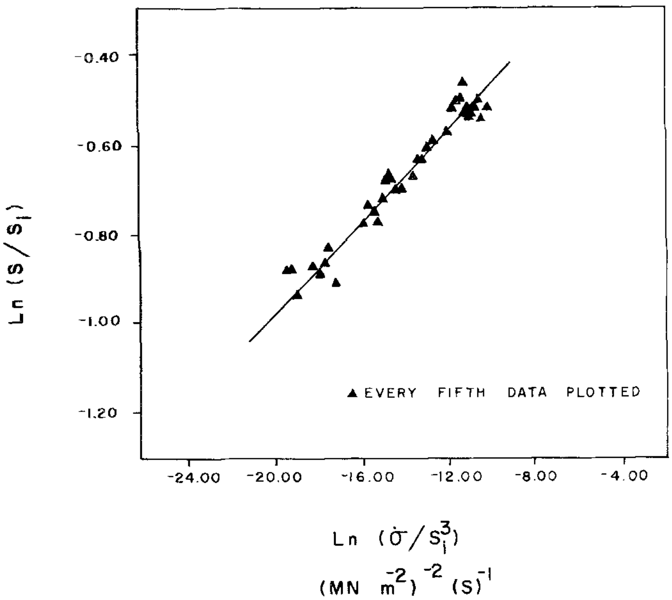



*Figure 6* Dynamic fatigue data of soda-lime glass tested in water at room temperature as analysed by the homologous stress technique. The regression line has a correlation coefficient of 0.96.

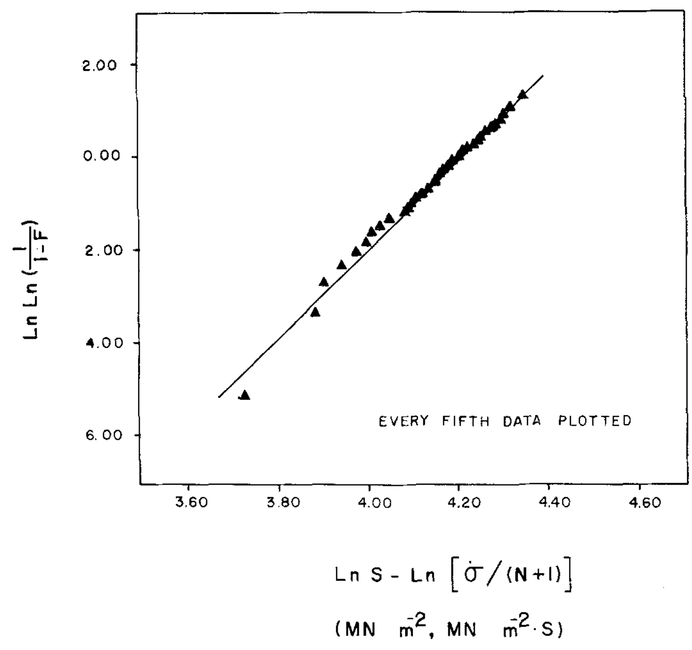



*Figure 7* Dynamic fatigue data of soda-lime glass tested in water at room temperature as analysed by the iterative, bivariant technique. The regression line has a correlation coefficient of 0.99.

evidenced by the smaller standard deviations. Furthermore, the excellent agreement between the Weibull slope, $m$, obtained from the iterative, trivariant analyses (7.88 and 7.95) with that obtained from the separate inert strength measurements (8.19) gives additional validity to this approach and to the fact that fatigue data contains information about not only the fatigue behaviour of the material in a given environment but also the initial inert strength distribution. In summary, it is believed that these results give general validity to the fracture mechanics framework for analysing fatigue data as presented in the preceding section of this paper.

To see more clearly that the differences in the fatigue parameters in Tables I and II do not lead to significant differences in failure predictions, the applied stress for a lifetime of $10^5$ sec at the cumulative failure probability of $10^{-3}$ was calculated using the fatigue parameters in Tables I and II.

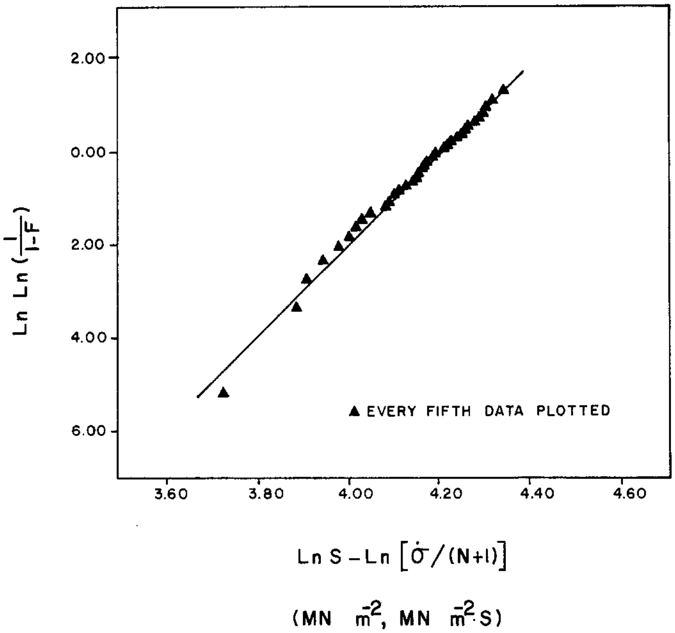



*Figure 8* Dynamic fatigue data of soda-lime glass tested in water at room temperature as analysed by the iterative, trivariant technique. The regression line has a correlation coefficient of 0.99.

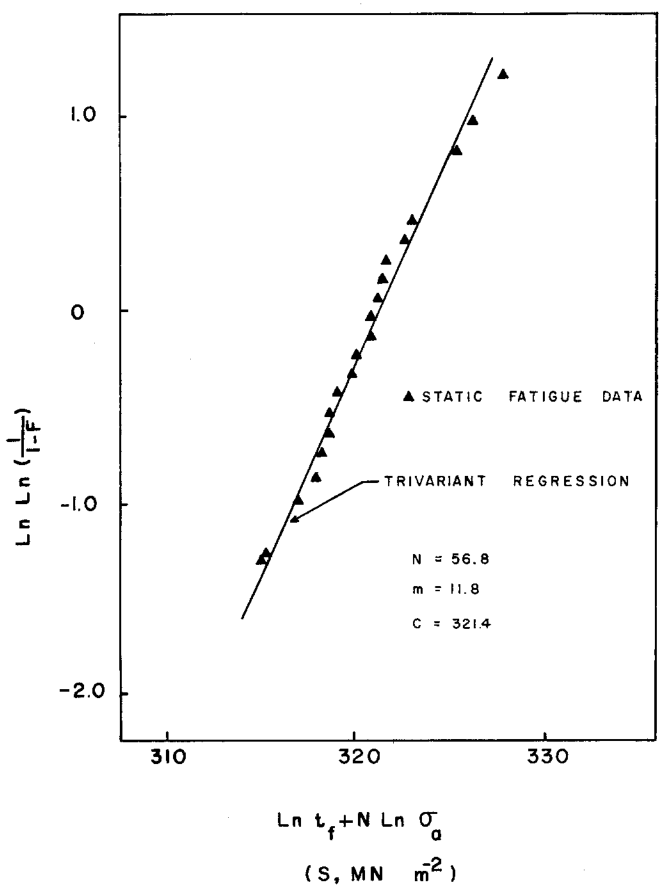



*Figure 9* Results of the iterative, trivariant analysis of the static fatigue data for as-fired, reaction-bonded silicon nitride tested in water-saturated air at room temperature. Data from [15]. The regression line has a correlation coefficient of 0.99.

TABLE III Predicted applied stress for a lifetime of $10^5$ sec at $F = 10^{-3}$, using the fatigue parameters given in Tables I and II

| Test data | Data analysis | $\sigma_a$ ($MN\,m^{-2}$) |
|---|---|---|
| Static fatigue | Median value | 19.61 (± 2.36)* |
| Static fatigue | Homologous stress | 18.15 (± .65) |
| Static fatigue | Iterative bivariant | 19.07 (± .67) |
| Static fatigue | Iterative Trivariant | 18.54 (± .65) |
| Dynamic fatigue | Median value | 18.54 (± 1.17) |
| Dynamic fatigeu | Homologous stress | 17.91 (± .71) |
| Dynamic fatigue | Iterative bivariant | 18.15 (± .64) |
| Dynamic fatigue | Iterative Trivariant | 17.64 (± .58) |

*The number in parentheses represents the 95% confidence limits.

Table III summarizes these calculations and it is seen that all of the predicted applied stresses are quite similar and are within the experimental accuracy of the data. Also included in Table III are the 95% confidence limits on the applied stress predictions. These confidence limits were calculated using the law of propagation of errors [6, 7]. As with the fatigue constants, it is quite evident that much greater confidence is obtained in the failure predictions based on the techniques that utilize all the data in the regression analysis.

The major advantage of the iterative techniques is that they do not require the testing of a large number of samples at each applied stress or stress rate condition since all the data are ranked together. This is most useful in situations where the testing of a large number of samples is not possible. With the iterative techniques it is not necessary to get all of the data at only a few applied stresses or stressing rates but rather a more "scatter shot" approach can be used whereby a large number of applied stresses or stressing rates are employed. The advantage of this "scatter shot" approach is that the data could be spread over a broad range of stresses or stressing rates. In addition, the iterative, trivariant analysis will be most useful in those cases where inert strength measurements may be impractical; yet, one would still like to make failure predictions based on fracture mechanics principles. To illustrate the usefulness of the iterative, trivariant technique static fatigue data [15] for 27 samples of as-fired, reaction-sintered $Si_3N_4$ tested at 5 different applied stress levels in water-saturated air at room temperature was analysed by this technique and the results are shown in Fig. 9. It is quite evident that the data fits the regression analysis exceptionally well. The values obtained for the constants in Equation 11 are $N = 56.8$, $m = 11.8$, and $C = 321.4$. Based on the fracture strength data in dry air for these same samples [15] the value obtained from $m$ appears quite reasonable. Also, the value for $N$ agrees quite well with that determined from crack velocity data on similar samples [16] where $N$ was found to be 59. Once the constants in Equation 11 have been determined, failure predictions can now be made for these $Si_3N_4$ samples. For example, for a lifetime of $10^5$ sec under an applied stress of 135 MN $m^{-2}$, the probability of failure is about $10^{-3}$. By decreasing the applied stress to 75 $MN\,m^{-2}$ for the same lifetime, reduced the failure probability to $10^{-6}$. It is important to emphasize that without the use of the iterative, trivariant analysis, failure predictions could not have been made based on the limited static fatigue data obtained on this material. However, it is just this type of limited data that is often obtained in prototype testing of actual components.

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